Example: Let's approximate $\int_0^4 \sqrt{x}\,dx$.

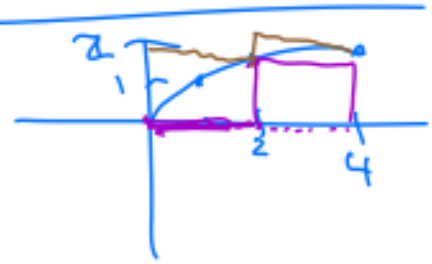
Exact Value: $\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}(4)^{3/2} - 0$

$x^{1/2} \to \frac{2}{3}x^{3/2}$

$4^{3/2} = (4^{1/2})^3 = 2^3 = 8$

$= \boxed{\frac{16}{3}} \approx 5.3333\ldots$

Let's compute Left(2)
Right(2)
Trap(2)
& errors.



$\text{Left}(2) = \sum_{k=0}^{\overset{n-1=1}{1}} f(x_k)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x$

$= 0\cdot 2 + \sqrt{2}\cdot 2 = \boxed{2\sqrt{2}}$

≈ 2.828

$\Delta x = \frac{b-a}{n} = \frac{4}{2} = 2$

$x_j = \underset{0}{a} + j\underset{2}{\Delta x} = 2j$

$f(x_k) = f(2k)\ \sqrt{2k}$

$f(x_0) = 0$

$f(x_1) = \sqrt{2}$

$f(x_2) = \sqrt{4} = 2$

$\text{Right}(2) = f(x_1)\Delta x + f(x_2)\Delta x$

$= \sqrt{2}(2) + 2\cdot(2) = \boxed{4 + 2\sqrt{2}}$

≈ 6.828

$\text{Trap}(2) = \frac{1}{2}\left[\text{Left}(2) + \text{Right}(2)\right] = \frac{1}{2}\left(2\sqrt{2} + 4 + 2\sqrt{2}\right)$

$= 2\sqrt{2} + 2$

≈ 4.828

since $f(x) = \sqrt{x}$ is concave down, we expect Trap(n) < Exact.

(BTW: Mid(n) would be too big.)

What about Left(n), Right(n), Trap(n)?

$$\text{Left}(n) = \sum_{k=0}^{n-1} f(x_k)\Delta x = 4\frac{}{n}\sum_{k=0}^{n-1}\sqrt{\frac{4k}{n}}$$

$\frac{4^{1/2}k^{1/2}}{n^{1/2}}$

$\Delta x = \frac{b-a}{n} = \frac{4}{n}$

$x_k = a + k\Delta x = 0 + k\frac{4}{n} = \frac{4k}{n}$

$f(x) = \sqrt{x}$

$$= \frac{8}{n}\sum_{k=0}^{n-1}\sqrt{\frac{k}{n}}$$

$$= \boxed{\frac{8}{n^{3/2}}\sum_{k=0}^{n-1}\sqrt{k}}$$

$$\text{Right}(n) = \cdots = \boxed{\frac{8}{n^{3/2}}\sum_{k=1}^{n}\sqrt{k}}$$

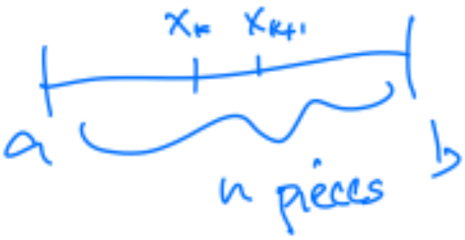
$\frac{\text{add}}{2} = \text{Trap.}$

$$\text{Trap}(n) = \frac{8}{n^{3/2}}\left(\underbrace{\tfrac{1}{2}\cdot\sqrt{0}}_{k=0} + \sum_{k=1}^{n-1}\sqrt{k} + \underbrace{\tfrac{1}{2}\sqrt{n}}_{k=n}\right)$$

$$= \boxed{\frac{8}{n^{3/2}}\sum_{k=1}^{n-1}\sqrt{k} + \frac{4}{n}}$$

Question: How do these #s depend on n, and how does the error depend on n?

Error in Left(N)



$f(x_k) = y_k$

degree 0 finite difference formula

$h = \Delta x = \frac{b-a}{n}$

$$\int_{x_0}^{x_1} f(x)\,dx =$$

$$= \int_{x_0}^{x_1} \left(y_0 + \frac{f'(c)}{1!}(x - x_0) \right) dx$$

c is between x & x_0

$$= \int_{x_0}^{x_1} y_0\,dx + \int_{x_0}^{x_1} \underbrace{f'(c)}_{\text{between the}} \underbrace{(x - x_0)}_{+}\,dx$$

$y_0 x \Big|_{x_0}^{x_1}$

between the $\max\{f'(x) : x_0 \le x \le x_1\} = M$

& $\min\{f'(x) : x_0 \le x \le x_1\} = m$

$$= y_0(x_1 - x_0) + \text{stuff.}$$

$$m \int_{x_0}^{x_1} (x - x_0)\,dx \le \text{stuff} \le M \int_{x_0}^{x_1} (x - x_0)\,dx$$

$$\left[\frac{(x - x_0)^2}{2}\right]_{x_0}^{x_1}$$

$$= \frac{(x_1 - x_0)^2}{2} - 0$$

$$= \frac{h^2}{2}$$

$x_0 = a$

$x_1 = a + \Delta x = a + h$

$x_1 - x_0 = h$

$$m\frac{h^2}{2} \le \text{stuff} \le M\frac{h^2}{2}$$

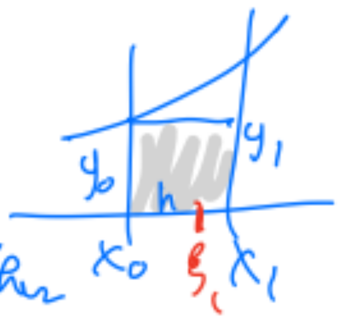
$$\Rightarrow \text{stuff} = f'(\xi)\frac{h^2}{2}$$

ξ some # between x_0 & x_1

using intermed value thm applied to $f'(x)$ (Must have $f'(x)$ is continuous).

Recap:

$$\int_{x_0}^{x_1} f(x)\,dx = \underbrace{y_0 h}_{\text{Left approx.}} + f'(\xi_1)\frac{h^2}{2}$$



Do the same for all the other intervals:

$$\int_{x_k}^{x_{k+1}} f(x)\,dx = y_k h + f'(\xi_{(k+1)})\frac{h^2}{2}$$

Add up from $k=0$ to $k=n-1$

$$\int_a^b f(x)\,dx = \underbrace{Left(n)}_{h(y_0+y_1+\dots+y_{n-1})} + \left(\sum_{k=0}^{n-1} f'(\xi_{k+1})\right)\frac{h^2}{2}$$

Average of $f'(\xi_k)$'s is $\frac{1}{n}\sum_{k=0}^{n-1}\left(f'(\xi_{k+1})\right) = \underbrace{f'(\tau)}_{\text{intermediate value thm}}$

τ is some # between a & b.

$$\Rightarrow \sum_{k=0}^{n-1} f'(\xi_{k+1}) = n f'(\tau).$$

$$\int_a^b f(x)\,dx = \text{Left}(n) + \frac{n f'(\tau) h^2}{2}$$

$$h = \frac{(b-a)}{n} \qquad \frac{n f'(\tau)\left(\frac{b-a}{n}\right)^2}{2}$$

$$\boxed{\int_a^b f(x)\,dx = \text{Left}(n) + \frac{f'(\tau)}{2n}(b-a)^2}$$

$$\boxed{\text{Error in Left}(n) = \frac{f'(\tau)}{2n}(b-a)^2}$$

where τ is some # between a & b.

Thus a bound on $|f'(x)|$ gives us a bounds on the error.

Similar calculation

$$\boxed{\text{Error in Right}(n) = \frac{-f'(\alpha)}{2n}(b-a)^2}$$

where α is some # between a & b.